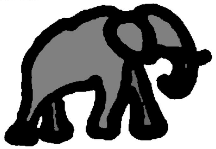
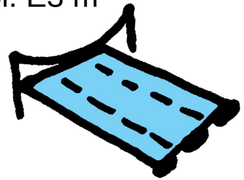
Order-of-Magnitude of Common Comparisons

Elephant:
E3 kg



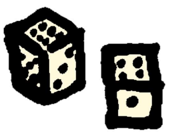
Olympic swimming pool: E3 m^3



Hairsbreadth:
E-4 m

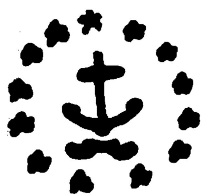
Blink of an eye:
E-1 s

One-in-a-million chance: E-6

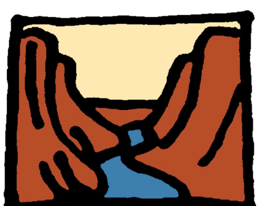


Grains of sand on Earth: E24

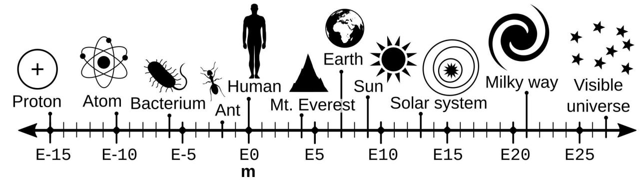
Area of Rhode Island:
E9 m^2



Grand Canyon:
E13 m^3



Thinking by Orders-of-Magnitude



What?

The order-of-magnitude of a value is simply how many times you have to multiply / divide by `10` to approximate it.
For `10` it is E1, for `1000` it is E3, for `0.01`: E-2 and so on.

We write it as ‘E3‘ as a shorthand (see p. 6)

Thinking by orders-of-magnitude simply means that, for example, instead of thinking of the Earth as `13,000` km (`1.3E7` m) across, you think of the order-of-magnitude: E7 m.

This has a few advantages:

- To memorize values you only have to remember a small number.
- You can multiply values just by adding their orders-of-magnitude.
- Most important: By keeping the values in your head, you can internalize the relationships between them.

What all can you do with this?

Getting a sense of scale

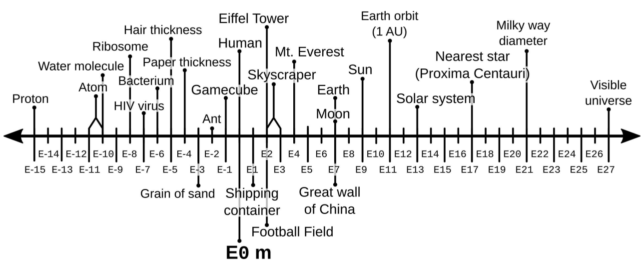
You can use this kind of thinking on all sorts of values (size, mass, time, volume, …). The first step is to be able to convert normal values to order-of-magnitude values. We'll go over that in the next few pages.

After that we'll get familiar with the different scales. The scale for size/extent is below, then later we see mass and time.

Lastly we'll look at some more applications and finish off with some useful resources.

Atom	E-10 m
Bacterium	E-6 m
Ant	E-2 m
Human	E0 m
Football Field	E2 m
Earth	E7 m

Sun	E9 m
Earth Orbit	E11 m
Nearest star	E17 m
Milky Way	E21 m
Visible universe	E27 m



Useful Resources

Homepage for this zine

Includes sources and calculations

https://nathanmcrae.name/oom-zine

The Physics Factbook

https://hypertextbook.com/facts/

Wikipedia Category Orders-of-Magnitude

https://en.wikipedia.org/wiki/Category:Orders_of_magnitude

Bionumbers Database

https://bionumbers.hms.harvard.edu/search.aspx

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Applications: Probability

Thinking of probability in terms of orders-of-magnitude is excellent for very small probabilities. A 1 in 10 probability is E-1, a one-in-a-million is E-6

Event	Prob.
Winning the lottery with one ticket (UK national lottery, 2009)	E-7
Yellowstone erupting in any given year	E-6
Being dealt a straight flush in poker	E-5
Being dealt four of a kind in poker	E-4
Sharing a birthday with any one random person	E-3
Giving birth to twins	E-2
Rolling a natural 20 on a D20.	E-1
Getting heads on a fair coin toss	E0

A basic rule is that the probability of two independent events both occurring is their probabilities multiplied. So, for example, the probability of winning the lottery and Yellowstone both erupting in a year is E-7 × E-6 ~ E-13.

Hopefully all these scales and applications give you a sense of how much depth and breadth order-of-magnitude thinking affords you. There's still a lot more to explore though.

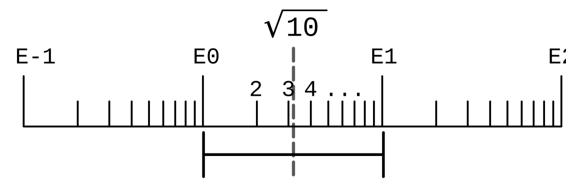
Here are some resources to get you started:

… It works like this:

The order-of-magnitude for numbers like 10, 10000, or 0.001 is straightforward (E1, E4, E-3), but it might not be so clear for numbers like 50, 32000, or 0.00465.

The idea is that the order-of-magnitude of a number is the power of 10 that best approximates it. Let's look at an example: *What is the order-of-magnitude of 40?*

So what is the power of 10 that best approximates 40? Clearly it should be either 10 or 100, but which one? 10 is closer to *40 (40 - 10 = 30, 100 - 40 = 60)*. But what we care about in order-of-magnitude thinking is **scale**, and the question is how different they are in scale: *40 / 10 = 4*, but *100 / 40 = 2.5.* This means 40 is closer to 100 in terms of scale and so 100 is the best order-of-magnitude approximation.



So what is the general rule here? Given two adjacent orders-of-magnitude, what is the midpoint between them in terms of scale? This is clearest in the case of 1 and 10: What is the number a such that $1 \times a = a$ and $a \times a = 10$? Well that's just $\sqrt{10}$: $\sqrt{10} \times \sqrt{10} = 10$. So the midpoint is $\sqrt{10} \cong 3.16$: any value above it is on the order-of-magnitude of 10, any value below it is on the order-of-magnitude of 1. This same reasoning applies to all powers of 10.

But let's try to simplify things...

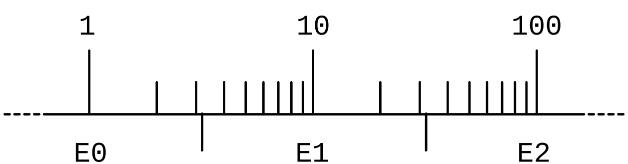
Keep it simple:

It's nice to know the reasoning behind the conversion to order-of-magnitude, but it's better to be able to remember a simple process:

1. Take the number (for example `53,000`) and move the most significant digit to the ones place (move 5 down 4 places: `5.3E4`).
2. If the resulting value is above `3.16` ($\sqrt{10}$), add `1` to the order-of-magnitude: (`53,000 ~ E5`).

Here are some more examples:

	Value	Reduced	>3.16	OOM
Housefly weight	20 mg	2.0E-5 kg	✖	E-5 kg
Human lifespan	24E8 s	2.4E9 s	✖	E9 s
Light-year	9.5E12 km	9.5E15 m	✔	E16 m
Baseball mass	5 oz.	1.4E-1 kg	✖	E-1 kg
Everest height	8,848.86 m	8.9E3 m	✔	E4 m



TIP: In a spreadsheet, if you want to calculate the order-of-magnitude of a value, you can use this formula: `=ROUND(LOG10(<value>), 0)`

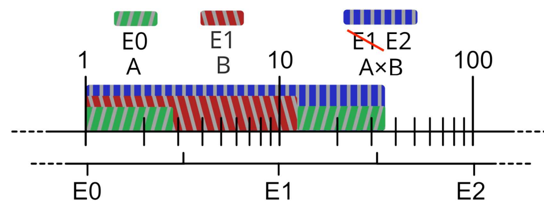
Aside: how do you manage errors?

Taking the order-of-magnitude of a value is an approximation and inevitably leads to approximation errors. Suppose you know the speed of light is E8 m/s and a year is E7 s, then a light year should be E15 m. However the more accurate calculation is

```
3E8 m/s × 3E7 s = 9E15 m ~ E16 m
```

With more experience it's possible to get a sense of when this will happen, but you can't avoid it entirely. So the question is what to do knowing this can occur.

The thing to remember is that order-of-magnitude values are better for memorization than for calculation. Order-of-magnitude arithmetic is a useful tool to have, but not something to use if you can't stand to be off by an order-of-magnitude here or there.



This means that if whatever value you've calculated is only E1 less than the value you're comparing against, that doesn't necessarily mean it's actually smaller. Employ appropriate skepticism for calculated values.

Applications: Data / Data Rate

We take the amount of data to be the number of bits required to represent it, and then the transmission rate is bits/s.

Data	Bits
Still image (1024x768)	E6
Average book / 1 Megabyte	E7
Complete works of Shakespeare	E8
One song (5 minutes, CD quality)	E9
Human genome / 1 Gigabyte	E10
Wikipedia text (compressed, 2013)	E11
Library of Congress (2005)	E16
All stored digital information (2009)	E22

So, how long would it take to transmit all of Wikipedia over a dialup connection?

E11 (bits) / E5 (bits / s) ~ E[11 - 5] s ~ E6 s

Data Rate	Bits / s
Low-quality phone	E4
Dial-up	E5
CD Audio	E6
DVD Video	E8
Fast internet	E9
Global internet traffic (2017)	E17

Order-of-Magnitude Arithmetic

One of the biggest advantages of using order-of-magnitude values, is that, to multiply the values, you can simply add the orders-of-magnitude. This is easiest to see in the case of multiplying by 10 which simply adds an order-of-magnitude e.g.

1000 × 10 = 10,000

(E3 × E1 ~ E[3 + 1] ~ E4)

This comes from a basic rule of exponents: $10^a \times 10^b = 10^{(a+b)}$.

Here's an example: Suppose we work at the toothpick factory where the toothpick machine makes 9 toothpicks per second. How many do we make in an 8 hour (28800 s) shift?

We have E1 toothpicks / s and E4 s, so multiplying them would be E[1 + 4] ~ E5 toothpicks.

To check, we can do the full calculation:

28800 s × 9 tp / s = 259200 ~ E5 tp

Since the order-of-magnitude calculations are just simple addition, it allows you to focus on the underlying relationships.

Aside: units

The choice of units to use is somewhat arbitrary since any unit will work (e.g. the milky way is E44 oz), but for the sake of consistency and to take advantage of some of it's special relationships (e.g. 1 m^3 of water = 1E3 kg), I use the SI base units.

Note: kg is the actual base unit of mass, rather than g. And since kg is slightly more common anyway, I suggest using that.

Aside: why do we write it like 'E2'?

Mainly we want a way to write orders-of-magnitude that distinguishes them from normal values. Often times you will see it written like '10^2', but the problem is that this is also just the normal number 100. To clarify, we have the awkward phrasing 'the order-of-magnitude of ten to the two'.

But we can borrow the notation '1E2' (often used on calculators and in programming) and just leave off the leading 1 to get 'E2'.

This makes it clear that it's not a normal value and can be thought of as saying 'The order-of-magnitude of 10^2' while being much shorter.

Along with this, we also use the '~' symbol to indicate that two values have the same order-of-magnitude. For example, an eyelash is 70 ng (E-7 kg) and a fruit fly is 200 ng (also E-7 kg). So eyelash mass ~ fruit fly mass.

Good shorthand can make thinking much easier by getting out of the way of common patterns and I think this particular notation accomplishes that well.

Before we get to more order-of-magnitude scales and applications of them, it might help to have an idea of what an order-of-magnitude looks like.

So what does an order-of-magnitude look like?

Application: Cost of Energy

How much does switching to LEDs save over the course of a few years?

To start with, electricity is E-7 $/J and LEDs are E1 W, while incandescents are E2 W (J/s). We want to know the cost rate i.e. $/s of the two, which would be (cost of electricity) × (energy usage).

This means LEDs are

E-7 $/J × E1 J/s ~ E[-7 + 1] $/s ~ E-6 $/s,

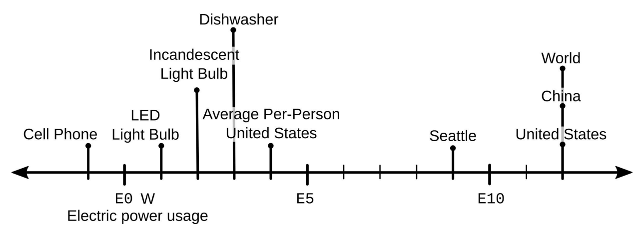
and incandescents are E[-7 + 2] $/s ~ E-5 $/s.

Calculating the cost for, let's say, 3 years ~ E8 s:

LEDs cost ~ E-6 $/s × E8 s ~ E2 $

Incandescents cost ~ E-5 $/s × E8 s ~ E3 $

It's interesting that the relative energy usage doesn't change with time. LEDs will always be E1 cheaper than incandescents. This is the kind of relationship that might be difficult to internalize normally, but really stands out when thinking with orders-of-magnitude.



Getting a sense of scale: Time

	OOM (s)
CPU cycle	E-10
Neuron firing	E-3
Human reflexes	E-2
Blink of an eye	E-1
Heartbeat	E0
Vine	E1
Minute	E2
Hour	E3
Day	E5
Month	E6
Year	E7

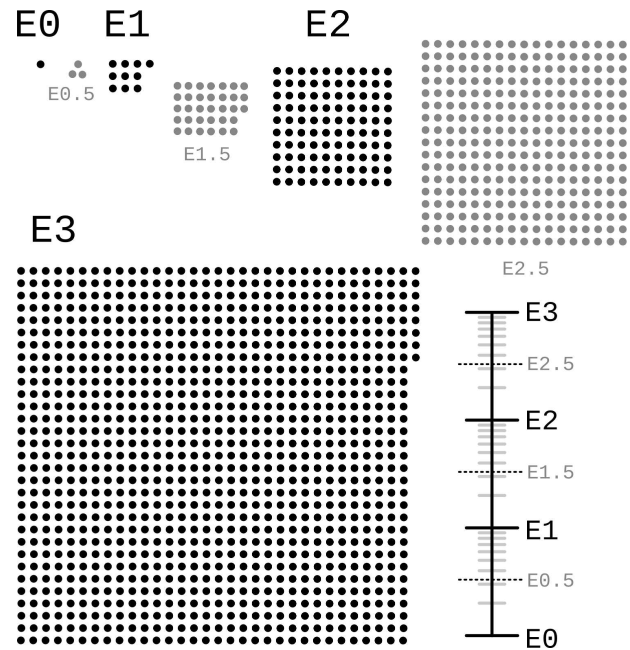
Note: The conversion between seconds and years is right on the boundary between E7 and E8. It's technically closer to E7, so that's what I use if necessary. But because of that, the conversion is more likely to lead to errors so I tend to avoid converting between them unless it's the only option.

Alternatively, we can see what each power of ten in seconds looks like to get a feeling for each order-of-magnitude.

1E2 s	1 1/2 minutes
1E3 s	17 minutes
1E4 s	2 hours 45 minutes
1E5 s	1 day
1E6 s	11 1/2 days
1E7 s	4 months
1E8 s	3 years

It looks like this:

Here we can see the black dots in groups of 1, 10, 100, and 1000. The groups of gray dots are halfway between the other groups (in terms of scale). You can see that the gray groups are about 3 times (or $\sqrt{10}$, more precisely) bigger than the groups smaller than them, and about 1/3 the size of the groups larger than them.



Mass by Order-of-Magnitude

Getting a sense of what an order-of-magnitude difference looks / feels like allows you to apply that intuition to many other relationships.

Suppose you have a good intuition for how large a housefly is compared to a fruit fly (at E2 times heavier). Then, knowing that the Earth is E2 times more massive than the Moon, we can use that same intuition to get a rough understanding of their relationship.

This is where internalizing a full scale like the one below becomes important. By having many of these small relationships at hand, it becomes easy to develop this intuition and apply it to new values as you see them.

With that in mind, we'll next look at the scale of time, and then go through some applications.

Aside: Common Units of Measure

Inch:	E-2 m	Fluid ounce:	E-5 m^3
Foot:	E-1 m	Teaspoon:	E-5 m^3
Mile:	E3 m	Tablespoon:	E-5 m^3
Astronomical Unit:	E11 m	Shot:	E-5 m^3
Light-year:	E16 m	Cup:	E-4 m^3
		Quart:	E-3 m^3
Ounce:	E-2 kg	Gallon:	E-2 m^3
Pound:	E0 kg	Cord:	E1 m^3
Acre:	E4 m^2	Mole:	E24 particles
Football field:	E4 m^2	Horsepower:	E3 W
Square mile:	E6 m^2	1 ton TNT:	E10 J